

WEAK MAASS-POINCARÉ SERIES AND WEIGHT 3/2 MOCK MODULAR FORMS

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ABSTRACT. The primary goal of this paper is to construct the basis of the space of weight $3/2$ mock modular forms which is an extension of the Borchers-Zagier basis of weight $3/2$ weakly holomorphic modular forms. The shadows of the members of this basis form the Borchers-Zagier basis of the space of weight $1/2$ weakly holomorphic modular forms. For the purpose, we use a weak Maass-Poincaré Series. The secondary goal is to provide a full computation of the Fourier coefficients for the weak Maass-Poincaré Series in most general form as a weak Maass-Poincaré Series has played a key role in the recent advances in the theory of weak Maass forms.

1. INTRODUCTION

A *weak Maass form* of weight k for the group $\Gamma_0(N)$ is a smooth function defined in the upper half plane which is invariant under the action of all elements in $\Gamma_0(N)$ and is an eigenform of the weight k hyperbolic Laplace operator. In addition, a weak Maass form has at most exponential growth at all cusps (See [7, Section 3]). We call a weak Maass form *harmonic* if it is annihilated by the hyperbolic Laplacian and call a harmonic weak Maass form a *weakly holomorphic modular form* if it is holomorphic in the upper half plane with possible poles at the cusps. In fact, by its definition, the image of a harmonic weak Maass form under the differential operator $\xi_k := 2iy^k \frac{\bar{\partial}}{\partial \bar{z}}$ is always a weakly holomorphic modular form of weight $2 - k$. For $k \in \frac{1}{2}\mathbb{Z}$, let $M_k^!(N)$ denote the infinite dimensional vector space of weight k weakly holomorphic modular forms on $\Gamma_0(N)$ and $M_k^!$ denote the subspace of $M_k^!(4)$, in which each form satisfies Kohnen's plus space condition, that is, its Fourier expansion has the form $\sum a(n)q^n$ where $a(n)$ is non-zero only for integers n satisfying $(-1)^{k-1/2}n \equiv 0, 1 \pmod{4}$. There are natural infinite bases $\{f_d | d \equiv 0, 1 \pmod{4}, d \leq 0\}$ and $\{g_D | D \equiv 0, 3 \pmod{4}, D < 0\}$ for $M_{\frac{1}{2}}^!$ and $M_{\frac{3}{2}}^!$, respectively, where f_d and g_D have Fourier developments of the forms

$$(1.1) \quad f_d(z) = q^d + \sum_{D < 0} A(D, d)q^{|D|} \in M_{\frac{1}{2}}^!$$

and

$$(1.2) \quad g_D(z) = q^D + \sum_{d \leq 0} B(D, d)q^{|d|} \in M_{\frac{3}{2}}^!$$

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where $q = \exp(2\pi iz)$ and $z = x + iy \in \mathbb{H}$, the upper half plane. Surprisingly, $A(D, d) = -B(D, d)$ as proved by Zagier in [19].

Recently, Duke, Imamoglu and Tóth [9] extended the basis $\{f_d | d \equiv 0, 1 \pmod{4}, d \leq 0\}$ for $M_{\frac{1}{2}}^!$ to a basis $\{h_d | d \equiv 0, 1 \pmod{4}\}$ for $H_{\frac{1}{2}}^!$, the space of weight $1/2$ *weakly harmonic Maass forms* on $\Gamma_0(4)$, which are harmonic weak Maass forms satisfying the plus space condition. They showed that $h_d = f_d$ for $d \leq 0$ and $\xi_{1/2} h_d = -2\sqrt{d} g_{-d}$ for $d > 0$.

In fact, every harmonic weak Maass form $h(z)$ is decomposed to $h(z) = h^+(z) + h^-(z)$, where $h^+(z)$ is holomorphic and $h^-(z)$ is non-holomorphic on \mathbb{H} . If h has weight k , following Zagier [20], we call the holomorphic part $h^+(z)$ a *mock modular form of weight k* and the image of the non-holomorphic part h^- under ξ_k the *shadow* of the mock modular form h^+ . Note that $\xi_k(h) = \xi_k(h^-)$. Hence in [9], Duke, Imamoglu and Tóth eventually constructed the weight $1/2$ mock modular form f_d with shadow g_{-d} for each $0 < d \equiv 0, 1 \pmod{4}$. Furthermore, they found quite a few of interesting properties of the coefficients of the mock modular form f_d such as a relation with cycle integrals of the j -function and with modular integrals having rational period functions.

A natural question at this point is whether there exists a mock modular form g_D having f_{-D} as its shadow for each $0 \leq D \equiv 0, 3 \pmod{4}$ and the duality relation between f_d ($d \leq 0$) and g_D ($D < 0$) is preserved for f_d ($d > 0$) and g_D ($D \geq 0$), if there is any g_D ($D \geq 0$). The existence of such mock modular forms is proved in [5, Theorem 1.1] without an explicit construction. The primary goal of this paper is thus to construct the desired mock modular forms g_D .

Theorem 1.1. *There exists a unique basis $\{g_D | D \equiv 0, 3 \pmod{4}, D \geq 0\}$ for the space of mock modular forms of weight $3/2$ satisfying that each g_D has shadow f_{-D} and has a Fourier expansion of the form*

$$(1.3) \quad g_D(z) = \sum_{n \geq 0} b(D, n) q^n.$$

The recent advances in the theory of weak Maass forms have come along with rapid development in areas of mock modular forms, traces of CM values, and other related subjects. In many of such work, weak Maass-Poincaré Series played a key role (for example, [3, 4, 6, 8, 9, 13, 15]). While we will discuss arithmetic properties of the coefficients of g_D ($D \geq 0$) in the subsequent paper [12], in the course of proving Theorem 1.1 we provide a full computation of the Fourier coefficients for the weak Maass-Poincaré Series in most general form for the purpose of future reference.

We let $H_k^!(N)$ denote the space of harmonic weak Maass forms for $\Gamma_0(N)$ and $H_k(N)$ denote its subspace, in which the image of each form under ξ_k is a cusp form. As usual, $M_k(N)$ and $S_k(N)$ denote the space of classical modular forms and cusp forms for $\Gamma_0(N)$, respectively.

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2. WHITTAKER FUNCTIONS

We first recall many properties of Whittaker functions, with which we construct weak Maass-Poincaré Series as in [3, 4, 6, 8, 9, 13, 15]. Whittaker functions $M_{\mu, \nu}(y)$ and $W_{\mu, \nu}(y)$ are as

defined in [1, ch.13], linearly independent solutions of the Whittaker differential equation

$$(2.1) \quad \frac{d^2 w}{dy^2} + \left(-\frac{1}{4} + \frac{\mu}{y} + \frac{\frac{1}{4} - \nu^2}{y^2} \right) w = 0.$$

These functions can be expressed in terms of confluent hypergeometric functions as

$$(2.2) \quad M_{\mu,\nu}(y) = e^{-y/2} y^{\nu+1/2} M(\nu - \mu + \frac{1}{2}, 1 + 2\nu, y)$$

$$(2.3) \quad W_{\mu,\nu}(y) = e^{-y/2} y^{\nu+1/2} U(\nu - \mu + \frac{1}{2}, 1 + 2\nu, y),$$

where

$$(2.4) \quad M(a, b, x) = \sum_{n=0}^{\infty} \frac{(a)(a+1)(a+2) \cdots (a+n-1)}{(b)(b+1)(b+2) \cdots (b+n-1)} \frac{x^n}{n!}$$

and

$$(2.5) \quad U(a, b, x) = \frac{\Gamma(1-b)}{\Gamma(a-b+1)} M(a, b, x) + \frac{\Gamma(b-1)}{\Gamma(a)} x^{1-b} M(a-b+1, 2-b, x).$$

Here $\Gamma(z)$ is the Gamma function. The equation (2.5) implies, if $2\nu \notin \mathbb{Z}$,

$$(2.6) \quad W_{\mu,\nu}(y) = \frac{\Gamma(-2\nu)}{\Gamma(\frac{1}{2} - \nu - \mu)} M_{\mu,\nu}(y) + \frac{\Gamma(2\nu)}{\Gamma(\frac{1}{2} + \nu - \mu)} M_{\mu,-\nu}(y)$$

and in particular, $W_{\mu,\nu}(y) = W_{\mu,-\nu}(y)$.

The Whittaker functions may have integral representations for certain fixed values of μ and ν . If $\operatorname{Re}(\nu \pm \mu + 1/2) > 0$, then

$$(2.7) \quad M_{\mu,\nu}(y) = y^{\nu+1/2} e^{y/2} \frac{\Gamma(1+2\nu)}{\Gamma(\nu + \mu + 1/2) \Gamma(\nu - \mu + 1/2)} \int_0^1 t^{\nu+\mu-1/2} (1-t)^{\nu-\mu-1/2} e^{-yt} dt$$

and if $\operatorname{Re}(\nu - \mu + 1/2) > 0$, then

$$(2.8) \quad W_{\mu,\nu}(y) = y^{\nu+1/2} e^{y/2} \frac{1}{\Gamma(\nu - \mu + 1/2)} \int_1^\infty t^{\nu+\mu-1/2} (t-1)^{\nu-\mu-1/2} e^{-yt} dt.$$

It follows from (2.7) and (2.8) that if $\nu - \mu = 1/2$, then we have

$$(2.9) \quad M_{\mu,\nu}(y) + (2\mu + 1) W_{\mu,\nu}(y) = \Gamma(2\mu + 2) y^{-\mu} e^{y/2},$$

and from (2.8) that if $\nu + \mu = 1/2$, we have

$$(2.10) \quad W_{\mu,\nu}(y) = y^\mu e^{-y/2}.$$

If $\frac{1}{2} - \mu \pm \nu$ is an integer, Whittaker functions can be expressed as an incomplete gamma function $\Gamma(a, x) = \int_x^\infty e^{-t} t^{a-1} dt$. For example, if $\nu \in \frac{1}{2}\mathbb{Z}$, then

$$(2.11) \quad W_{\nu-1/2,\nu}(y) = e^{y/2} y^{1/2-\nu} \Gamma(2\nu, y).$$

Asymptotic behavior of the Whittaker functions for fixed μ, ν is also found from (2.7) and (2.8):

$$(2.12) \quad M_{\mu,\nu}(y) \sim \frac{\Gamma(1+2\nu)}{\Gamma(\nu - \mu + 1/2)} y^{-\mu} e^{y/2} \quad \text{and} \quad W_{\mu,\nu}(y) \sim y^\mu e^{-y/2} \quad \text{as } y \rightarrow \infty,$$

$$(2.13) \quad M_{\mu,\nu}(y) \sim y^{\nu+1/2} \quad \text{and} \quad W_{\mu,\nu}(y) \sim \frac{\Gamma(2\nu)}{\Gamma(\nu - \mu + 1/2)} y^{-\nu+1/2} \quad \text{as } y \rightarrow 0$$

Now we define for fixed value $s \in \mathbb{C}$ and $n \in \mathbb{Z}$,

$$(2.14) \quad \mathcal{M}_{n,k}(y, s) = \begin{cases} \Gamma(2s)^{-1} (4\pi|n|y)^{-k/2} M_{\frac{k}{2}\text{sgn}(n), s-1/2}(4\pi|n|y), & \text{if } n \neq 0, \\ y^{s-k/2}, & \text{if } n = 0. \end{cases}$$

$$\mathcal{W}_{n,k}(y, s) = \begin{cases} \Gamma(s + \frac{k}{2}\text{sgn}(n))^{-1} |n|^{k/2-1} (4\pi y)^{-k/2} W_{\frac{k}{2}\text{sgn}(n), s-1/2}(4\pi|n|y), & \text{if } n \neq 0, \\ \frac{(4\pi)^{1-k} y^{1-s-k/2}}{(2s-1)\Gamma(s-k/2)\Gamma(s+k/2)}, & \text{if } n = 0. \end{cases}$$

These are generalizations of the functions treated in [6, Chapter 1.3], [8, Section 3] and [9, Section 2] preserving important properties of them. In particular, if $e(z) = \exp(2\pi iz)$, then the function

$$(2.15) \quad \varphi_{m,k}(z, s) := \mathcal{M}_{m,k}(y, s) e(mx)$$

is an eigenfunction of the weight k hyperbolic Laplacian

$$\Delta_k = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + iky \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)$$

and has eigenvalue $s(1-s) + (k^2 - 2k)/4$ by (2.1). That is,

$$(2.16) \quad \Delta_k \varphi_{m,k}(z, s) = \left(s - \frac{k}{2} \right) \left(1 - \frac{k}{2} - s \right) \varphi_{m,k}(z, s).$$

Also, due to the asymptotic behavior of the Whittaker function given in (2.13),

$$(2.17) \quad \varphi_{m,k}(z, s) = O(y^{\text{Re}(s)-k/2}) \quad \text{as } y \rightarrow 0.$$

We are interested in the values of $s = k/2$ and $s = 1 - k/2$, for which $\Delta_k \varphi_{m,k}(z, s) = 0$. It follows from (2.14) and (2.10) for the case $n > 0$, from (2.14) and (2.8) for the case $n < 0$ and from (2.14) for the case $n = 0$ that

$$(2.18) \quad \mathcal{W}_{n,k}(y, 1 - k/2) = e^{-2\pi ny} \begin{cases} n^{k-1}, & \text{if } n > 0, \\ |n|^{k-1} \Gamma(1-k)^{-1} \Gamma(1-k, -4\pi ny), & \text{if } n < 0, \\ \frac{(4\pi)^{1-k}}{\Gamma(2-k)}, & \text{if } n = 0. \end{cases}$$

If we assume $k \leq 1/2$, then by using (2.14) and (2.7), we find that

$$(2.19) \quad \mathcal{M}_{n,k}(y, 1 - k/2) = e^{-2\pi ny} \begin{cases} (-1)^k [\Gamma(1-k)^{-1} \Gamma(1-k, -4\pi ny) - 1], & \text{if } n > 0, \\ 1 - \Gamma(1-k)^{-1} \Gamma(1-k, -4\pi ny), & \text{if } n < 0, \\ y^{1-k}, & \text{if } n = 0. \end{cases}$$

Also, using the fact $W_{\mu,\nu}(y) = W_{\mu,-\nu}(y)$ and (2.10) for the case $n > 0$ and directly from (2.14) for other cases, we have

$$(2.20) \quad \mathcal{W}_{n,k}(y, k/2) = e^{-2\pi ny} \begin{cases} \Gamma(k)^{-1} n^{k-1}, & \text{if } n > 0, \\ 0, & \text{if } n \leq 0. \end{cases}$$

Finally, using (2.14) and (2.2), we obtain

$$(2.21) \quad \mathcal{M}_{n,k}(y, k/2) = e^{-2\pi ny} \begin{cases} \Gamma(k)^{-1}, & \text{if } n \neq 0, \\ 1, & \text{if } n = 0. \end{cases}$$

3. WEAK MAASS-POINCARÉ SERIES

Consider the group

$$\mathfrak{G} := \{(A, \phi(z)) \mid A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{R}), \phi: \mathbb{H} \rightarrow \mathbb{C} \text{ hol}, |\phi(z)| = (\det A)^{-1/4} |cz + d|^{1/2}\}$$

with group law $(A_1, \phi_1(z))(A_2, \phi_2(z)) = (A_1 A_2, \phi_1(A_2 z) \phi_2(z))$. \mathfrak{G} acts on complex valued functions f defined on the upper half plane by $(f|_k(A, \phi)(z)) = \phi(z)^{-2k} f(Az)$. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$, define $j(\gamma, z)$ by

$$j(\gamma, z) = \begin{cases} \sqrt{cz + d}, & \text{if } k \in \mathbb{Z}, \\ \left(\frac{c}{d}\right) \varepsilon_d^{-1} \sqrt{cz + d}, & \text{if } k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}, \end{cases}$$

where \sqrt{z} is the principal branch of the holomorphic square root and

$$\varepsilon_d := \begin{cases} 1, & \text{if } d \equiv 1 \pmod{4}, \\ i, & \text{if } d \equiv 3 \pmod{4}. \end{cases}$$

The map $\Gamma_0(N) \rightarrow \mathfrak{G}$ defined by $\gamma \mapsto \tilde{\gamma} = (\gamma, j(\gamma, z))$ is a group homomorphism. For convenience, we write $f|_k \gamma$ instead of $f|_k \tilde{\gamma}$. Namely, the weight k slash operator is given by

$$(f|_k \gamma)(z) = j(\gamma, z)^{-2k} f(\gamma z).$$

If $\varphi := \varphi_{m,k}(z, s)$ is the function defined in (2.15) and $\Gamma_\infty = \{\pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \mid n \in \mathbb{Z}\}$ is the subgroup of translations of $\Gamma_0(N)$, then the Poincaré series

$$(3.1) \quad F_{m,k,N}(z, s) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} (\varphi|_k \gamma)(z)$$

is $\Gamma_0(N)$ -invariant and it follows from (2.17) that $F_{m,k,N}(z, s)$ converges absolutely and uniformly on compacta for $\text{Re}(s) > 1$. Moreover, $F_{m,k,N}(z, s)$ is an eigenfunction of the Laplacian Δ_k . In proving these, we use the same arguments with those in [7] and [9, p. 8].

Lemma 3.1. *Let $m \in \mathbb{Z}$ and $s \in \mathbb{C}$. The Poincaré series $F_{m,k,N}(z, s)$ converges absolutely and uniformly on compacta for $\text{Re}(s) > 1$, and it is a $\Gamma_0(N)$ -invariant eigenfunction of the Laplacian Δ_k satisfying*

$$(3.2) \quad \Delta_k F_{m,k,N}(z, s) = \left(s - \frac{k}{2}\right) \left(1 - \frac{k}{2} - s\right) F_{m,k,N}(z, s).$$

Thus $F_{m,k,N}(z, s)$ is real analytic.

Proof. By [4, Lemma 3.1] and (2.17), if $k > 2 - 2(\text{Re}(s) - k/2)$, then $F_{m,k,N}(z, s)$ is convergent and $\Gamma_0(N)$ -invariant. On the other hand, the Laplace operator Δ_k can be expressed in terms of the differential operator ξ_k

$$(3.3) \quad \Delta_k = -\xi_{2-k} \circ \xi_k.$$

The operator ξ_k and the slash operator commute under the rule $\xi_k(f|_k g(z)) = (\xi_k f)|_{2-k}(gz)$. Hence if we write $F := F_{m,k,N}(z, s)$ and $\Delta := \Delta_k$, then

$$\begin{aligned} \Delta F &= \Delta \sum (\varphi|_k \gamma) = \sum (-\xi_{2-k} \circ \xi_k)(\varphi|_k \gamma) = - \sum \xi_{2-k}(\xi_k(\varphi|_k \gamma)) \\ &= - \sum \xi_{2-k}(\xi_k(\varphi)|_{2-k} \gamma) = \sum (\Delta \varphi)|_k \gamma = (s(1-s) + (k^2 - 2k)/4)F, \end{aligned}$$

where the last equality follows from (2.16). \square

The function $F_{0,1/2,4}(z, s)$ is the usual weight $1/2$ Eisenstein series and $F_{m,1/2,4}(z, s)$ for any ineger m is briefly discussed in [9, Lemma 2]. Earlier, the Fourier expansion of the Poincaré series were computed for negative integer m and negative weight k in [6, Theorem 1.9] and for negative integer m and weight $k \geq 2$ in [8, Proposition 3.2]. We now provide the Fourier expansion of $F_{m,k,N}(z, s)$ for any integer m , arbitrary weight k and level N .

Theorem 3.2. *If m is an integer, then the Poincaré series $F_{m,k,N}(z, s)$ has the Fourier expansion*

$$F_{m,k,N}(z, s) = \mathcal{M}_{m,k}(y, s)e(mx) + \sum_{n \in \mathbb{Z}} c_{m,k}(n, s) \mathcal{W}_{n,k}(y, s)e(nx),$$

where the coefficients $c_{m,k}(n, s)$ are given by

$$(2\pi i)^{-k} \sum_{c>0} \frac{K_k(m, n, Nc)}{Nc} \times \begin{cases} |mn|^{\frac{1-k}{2}} J_{2s-1} \left(\frac{4\pi \sqrt{|mn|}}{Nc} \right), & mn > 0, \\ |mn|^{\frac{1-k}{2}} I_{2s-1} \left(\frac{4\pi \sqrt{|mn|}}{Nc} \right), & mn < 0, \\ 2^{k-1} \pi^{s+\frac{k}{2}-1} |m+n|^{s-\frac{k}{2}} (Nc)^{1-2s}, & mn = 0, m+n \neq 0, \\ 2^{2k-2} \pi^{k-1} \Gamma(2s) (2Nc)^{1-2s}, & m = n = 0, \end{cases}$$

where J_α and I_α are the usual Bessel functions as defined in [1, Ch. 9] and $K_k(m, n, c)$ is the (generalized) Kloosterman sum defined by

$$(3.4) \quad K_k(m, n, c) := \begin{cases} \sum_{v(c)^*} e\left(\frac{m\bar{v}+nv}{c}\right), & \text{if } k \in \mathbb{Z}, \\ \sum_{v(c)^*} \left(\frac{c}{v}\right)^{2k} \varepsilon_v^{2k} e\left(\frac{m\bar{v}+nv}{c}\right), & \text{if } k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}. \end{cases}$$

Here the sum runs through the primitive residue classes modulo c and $v\bar{v} \equiv 1 \pmod{c}$.

Proof. By the exactly same argument in the proof of Proposition 4 of [14], we have

$$F_{m,k,N}(z, s) = \mathcal{M}_{m,k}(y, s)e(mx) + \sum_{c>0, d(Nc)^*} C(c, d, N) \sum_{r \in \mathbb{Z}} (Nc(z+r) + d)^{-k} \varphi_{m,k} \left(\frac{a(z+r) + b}{Nc(z+r) + d}, s \right),$$

where $C(c, d, N) = 1$ if k is an integer and $C(c, d, N) = \left(\frac{Nc}{d}\right) \left(\frac{-4}{d}\right)^k$ if k is not an integer but a half integer. Now the theorem follows from Lemma 3.7 below. \square

It follows from (3.2) that $\Delta_k F_{m,k,N}(z, s) = 0$ for the special s values $s = k/2$ and $s = 1 - k/2$. If $k \leq 1/2$, then $F_{m,k,N}(z, s)$ is holomorphic in s near $s = 1 - k/2$ and if $k \geq 3/2$, then $F_{m,k,N}(z, s)$ is holomorphic in s near $s = k/2$. In fact, $F_{m,k,N}(z, k/2)$ is a weakly holomorphic modular form if $k > 2$ and $F_{m,k,N}(z, 1 - k/2)$ is a weak Maass form if $k < 0$.

Corollary 3.3. *If $k > 2$ and m is an integer, then the Fourier expansion of the Poincaré series $F_{m,k,N}(z, k/2)$ is given by*

$$F_{m,k,N}(z, k/2) = \frac{\delta_m}{\Gamma(k)} q^m + \sum_{0 < n \in \mathbb{Z}} \frac{c_{m,k}(n, k/2)}{\Gamma(k)} n^{k-1} q^n \in \begin{cases} S_k(N), & \text{if } m > 0, \\ M_k^1(N), & \text{if } m < 0, \\ M_k(N), & \text{if } m = 0. \end{cases}$$

where $\delta_m = \Gamma(k)$ if $m = 0$ and $\delta_m = 1$ otherwise and the coefficients $c_{m,k}(n, s)$ are defined in Theorem 3.2.

Proof. This is an immediate consequence of Theorem 3.2, (2.20) and (2.21). \square

The Fourier expansion above for negative m was computed in [8, Theorem 3.3] with slightly different normalization. Earlier, it was established for negative m and negative k in [6, Proposition 1.10] with slightly different setting. The following is its generalization into the case for every integer m and any level N .

Corollary 3.4. *If $k < 0$ and m is an integer, then the Poincaré series $F_{m,k,N}(z, 1 - k/2) \in H_k(N)$ if $m \leq 0$ and $F_{m,k,N}(z, 1 - k/2) \in H_k^!(N)$ if $m > 0$. Its Fourier expansion is given by*

$$F_{m,k,N}(z, 1 - k/2) = \mathfrak{m}_{m,k}(y)q^m + \frac{(4\pi)^{1-k}}{(1-k)\Gamma(1-k)}c_{m,k}(0, 1 - k/2) \\ + \sum_{n>0} c_{m,k}(n, 1 - k/2)n^{k-1}q^n + \sum_{n<0} c_{m,k}(n, 1 - k/2)|n|^{k-1}\frac{\Gamma(1-k, -4\pi ny)}{\Gamma(1-k)}q^n,$$

where

$$\mathfrak{m}_{m,k}(y) := \begin{cases} y^{1-k} & \text{if } m = 0, \\ 1 - \frac{\Gamma(1-k, -4\pi my)}{\Gamma(1-k)}, & \text{if } m < 0, \\ (-1)^{k-1} \left[1 - \frac{\Gamma(1-k, -4\pi my)}{\Gamma(1-k)} \right], & \text{if } m > 0 \end{cases}$$

and the coefficients $c_{m,k}(n, s)$ are defined in Theorem 3.2.

Proof. This result follows from Theorem 3.2, (2.18) and (2.19). \square

Many important properties of the weak Maass Poincaré series when $k < 0$ and $k > 2$ are discussed in [2]. For weights $0 \leq k \leq 2$, the Poincaré series does not converge. The theory of the resolvent kernel, however, assures that the Poincaré series $F_{m,k,N}(z, s)$ has an analytic continuation in s to $\text{Re}(s) > 1/2$ except for possibly finitely many simple poles in $(1/2, 1)$. These poles may only occur at points of the discrete spectrum of Δ_k . (Refer to [8, p. 386], [9, p. 12], [10, Section 3], [11].) Hence Theorem 3.2 gives Fourier expansions for the Maass Poincaré series of these weights as well. The weak Maass Poincaré series $F_{0,2,N}(z, s)$ and $F_{0,0,N}(z, s)$ are both usual Eisenstein series while $F_{m,0,1}(z, s)$ is the Niebur-Poincaré series which is a weak Maass form [9, 10, 16, 17]. Furthermore, the harmonic weak Maass Poincaré series $F_{m,0,1}(z, 1) = \overline{j_m(z)} + 24\sigma(m)$ where $j_m(z) = q^{-m} + O(q)$ ($m \geq 0$) form a unique basis for $\mathbb{C}[j]$. The Poincaré series $F_{m,2,1}(z, 1)$ is a weakly holomorphic modular form ([8, Theorem 3.3]) if $m < 0$ but it becomes trivial if $m > 0$. In [18], Tóth instead works on the derivative of $F_{m,2,1}(z, s)$ at $s = 1$ which is a harmonic weak Maass form. As half-integral weight forms have a better arithmetic meaning in the plus space, we discuss weak Maass forms of weight $1/2$ and $3/2$ satisfying the plus space condition in the next two sections.

Before we end this section, we prove Lemma 3.7 to complete the proof of Theorem 3.2. This approach is well described in Kohnen's earlier work [14] and we only generalize it to arbitrary weight k and arbitrary level N . Although the same method is used in many papers, none of them provide a full computation. We first state two results of Hejhal [11, Lemma 5.5, p.357] to prove Lemma 3.7.

Proposition 3.5 (Hejhal).

$$\int_{-\infty}^{\infty} (1-iu)^{-a}(1+iu)^{-b}e^{ixu}du = \begin{cases} \frac{\pi 2^{-\frac{a+b}{2}+1} x^{\frac{a+b}{2}-1}}{\Gamma(b)} W_{-\frac{a+b}{2}, \frac{a+b-1}{2}}(2x), & x > 0, \\ \pi 2^{2-a-b} \frac{\Gamma(a+b-1)}{\Gamma(a)\Gamma(b)}, & x = 0. \end{cases}$$

Proposition 3.6 (Hejhal).

$$\begin{aligned} \int_{-\infty}^{\infty} e^{2\pi i c \frac{u}{1+u^2}} e^{2\pi i \alpha u} \left(\frac{1-iu}{1+iu} \right)^l \frac{M_{l, s-\frac{1}{2}} \left(\frac{4\pi c}{1+u^2} \right)}{\Gamma(2s)} du \\ = \begin{cases} \frac{2\pi}{\Gamma(s+l)} W_{l, s-\frac{1}{2}}(4\pi\alpha) \frac{\sqrt{c}}{\sqrt{\alpha}} J_{2s-1}(4\pi\sqrt{\alpha c}), & \alpha > 0, \\ \frac{2\pi^{s+1}}{(s-\frac{1}{2})\Gamma(s-l)\Gamma(s+l)} c^s, & \alpha = 0, \\ \frac{2\pi}{\Gamma(s-l)} W_{-l, s-\frac{1}{2}}(4\pi\alpha) \frac{\sqrt{c}}{\sqrt{|\alpha|}} I_{2s-1}(4\pi\sqrt{|\alpha|c}), & \alpha < 0. \end{cases} \end{aligned}$$

Using these, we then derive a generalization of [14, Lemma, p. 253] and [9, Lemma 2] for any k and N .

Lemma 3.7. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$ have $c > 0$ and $\nu \in \mathbb{N}$ and suppose that $\text{Re}(s) > 1/2$. Then for $\varphi_{m,k}(z, s)$ defined in (2.15) with any $m \in \mathbb{Z}$, we have

$$\begin{aligned} \sum_{r \in \mathbb{Z}} (c(z+r) + d)^{-k} \varphi_{m,k} \left(\nu \frac{a(z+r) + b}{c(z+r) + d}, s \right) &= 2\pi i^{-k} \sum_{n \in \mathbb{Z}} e \left(\frac{a\nu m + nd}{c} \right) \mathcal{W}_{n,k}(y, s) e(nx) \\ &\times \begin{cases} |\nu mn|^{\frac{1-k}{2}} c^{-1} J_{2s-1} \left(\frac{4\pi \sqrt{|\nu mn|}}{c} \right), & \text{if } mn > 0, \\ |\nu mn|^{\frac{1-k}{2}} c^{-1} I_{2s-1} \left(\frac{4\pi \sqrt{|\nu mn|}}{c} \right), & \text{if } mn < 0, \\ 2^{k-1} \pi^{s+k/2-1} |\nu(m+n)|^{s-k/2} c^{-2s}, & \text{if } mn = 0, m+n \neq 0, \\ 2^{2k-1} \pi^{k-1} \Gamma(2s) (2c)^{-2s} \nu^{s-k/2}, & \text{if } m = n = 0. \end{cases} \end{aligned}$$

Proof. As explained in [14], it suffices to treat the case $M = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Since the series $\sum_{r \in \mathbb{Z}} (z+r)^{-k} \varphi_{m,k} \left(-\frac{\nu}{z+r}, s \right)$ converges absolutely uniformly on compact subsets of \mathbb{H} for $\text{Re}(s) > 1/2$, we may write

$$\sum_{r \in \mathbb{Z}} (z+r)^{-k} \varphi_{m,k} \left(-\frac{\nu}{z+r}, s \right) = \sum_{n \in \mathbb{Z}} \beta(n, y, s) e(nx).$$

For nonzero $m \in \mathbb{Z}$,

$$\begin{aligned}
\beta(n, y, s) &= \int_0^1 e(-nx) \sum_{r \in \mathbb{Z}} (z+r)^{-k} \varphi_{m,k} \left(-\frac{\nu}{z+r}, s \right) dx \\
&= \int_{-\infty}^{\infty} e(-nx) z^{-k} \varphi_{m,k} \left(-\frac{\nu}{z}, s \right) dx \\
&= \int_{-\infty}^{\infty} z^{-k} \left(\frac{4\pi\nu|m|y}{x^2+y^2} \right)^{-\frac{k}{2}} \frac{M_{\text{sgn}(m)\frac{k}{2}, s-\frac{1}{2}} \left(\frac{4\pi\nu|m|y}{x^2+y^2} \right)}{\Gamma(2s)} e \left(-\frac{\nu mx}{x^2+y^2} - nx \right) dx \\
&= (4\pi\nu|m|y)^{-\frac{k}{2}} i^{-k} I,
\end{aligned}$$

where we set

$$I = \int_{-\infty}^{\infty} \left(\frac{y-ix}{y+ix} \right)^{-\frac{k}{2}} \frac{M_{\text{sgn}(m)\frac{k}{2}, s-\frac{1}{2}} \left(\frac{4\pi\nu|m|y}{x^2+y^2} \right)}{\Gamma(2s)} e \left(-\frac{\nu mx}{x^2+y^2} - nx \right) dx.$$

Substituting $x = -\text{sgn}(m)yu$, we have

$$I = y \int_{-\infty}^{\infty} \left(\frac{1-iu}{1+iu} \right)^{\text{sgn}(m)\frac{k}{2}} \frac{M_{\text{sgn}(m)\frac{k}{2}, s-\frac{1}{2}} \left(\frac{4\pi\nu|m|}{y(u^2+1)} \right)}{\Gamma(2s)} e \left(\frac{\nu|m|u}{y(u^2+1)} + \text{sgn}(m)nyu \right) du.$$

Then setting $A = \text{sgn}(m)ny$ and $B = \frac{\nu|m|}{y}$, we get

$$I = y \int_{-\infty}^{\infty} \left(\frac{1-iu}{1+iu} \right)^{\text{sgn}(m)\frac{k}{2}} \frac{M_{\text{sgn}(m)\frac{k}{2}, s-\frac{1}{2}} \left(\frac{4\pi B}{u^2+1} \right)}{\Gamma(2s)} e \left(\frac{Bu}{u^2+1} + Au \right) du$$

so that we may apply Proposition 3.6 to obtain that

$$I = y \begin{cases} \frac{2\pi}{\Gamma(s+\text{sgn}(m)\frac{k}{2})} W_{\text{sgn}(m)\frac{k}{2}, s-\frac{1}{2}}(4\pi A) \sqrt{\frac{B}{A}} J_{2s-1}(4\pi\sqrt{AB}), & A > 0, \\ \frac{4\pi^{1+s}}{(2s-1)\Gamma(s+\frac{k}{2})\Gamma(s-\frac{k}{2})} B^s, & A = 0, \\ \frac{2\pi}{\Gamma(s-\text{sgn}(m)\frac{k}{2})} W_{-\text{sgn}(m)\frac{k}{2}, s-\frac{1}{2}}(4\pi|A|) \sqrt{\frac{B}{|A|}} I_{2s-1}(4\pi\sqrt{|A|B}), & A < 0. \end{cases}$$

Hence we have

$$\beta(n, y, s) = 2\pi i^{-k} \mathcal{W}_{n,k}(y, s) \begin{cases} |\nu nm|^{\frac{1-k}{2}} J_{2s-1}(4\pi\sqrt{\nu|mn|}), & mn > 0, \\ 2^{k-1} \pi^{s+\frac{k}{2}-1} |\nu m|^{s-\frac{k}{2}}, & mn = 0, \\ |\nu nm|^{\frac{1-k}{2}} I_{2s-1}(4\pi\sqrt{\nu|mn|}), & mn < 0. \end{cases}$$

If $m = 0$, then

$$\begin{aligned}
\beta(n, y, s) &= \int_0^1 e(-nx) \sum_{r \in \mathbb{Z}} (z+r)^{-k} \varphi_{0,k} \left(-\frac{\nu}{z+r}, s \right) dx \\
&= \int_{-\infty}^{\infty} e(-nx) z^{-k} \varphi_{0,k} \left(-\frac{\nu}{z}, s \right) dx \\
&= \int_{-\infty}^{\infty} e(-nx) z^{-k} \left(\frac{\nu y}{x^2 + y^2} \right)^{s-\frac{k}{2}} dx \\
&= (\nu y)^{s-\frac{k}{2}} \int_{-\infty}^{\infty} z^{-(s+\frac{k}{2})} \bar{z}^{-(s-\frac{k}{2})} e(-nx) dx.
\end{aligned}$$

By letting $x = yu$ and $x = -yu$ when $n \leq 0$ and $n > 0$, respectively, we have

$$\beta(n, y, s) = \nu^{s-\frac{k}{2}} y^{-s-\frac{k}{2}+1} i^{-k} \int_{-\infty}^{\infty} (1-iu)^{-(s+\frac{k}{2})} (1+iu)^{-(s-\frac{k}{2})} e^{2\pi i |n| y u} du.$$

If $n \neq 0$, then by Proposition 3.5,

$$\begin{aligned}
\beta(n, y, s) &= \nu^{s-\frac{k}{2}} y^{-s-\frac{k}{2}+1} i^{-k} \pi 2^{-s+1} (2\pi |n| y)^{s-1} \frac{W_{\text{sgn}(n)\frac{k}{2}, s-\frac{1}{2}}(4\pi |n| y)}{\Gamma(s + \text{sgn}(n)\frac{k}{2})} \\
&= 2\pi i^{-k} \mathcal{W}_{n,k}(y, s) 2^{k-1} \pi^{s+\frac{k}{2}-1} |\nu n|^{s-\frac{k}{2}}.
\end{aligned}$$

If $n = 0$, then by Proposition 3.5 again,

$$\begin{aligned}
\beta(n, y, s) &= \nu^{s-\frac{k}{2}} y^{-s-\frac{k}{2}+1} i^{-k} \pi 2^{2-2s} \frac{\Gamma(2s-1)}{\Gamma(s+\frac{k}{2}) \Gamma(s-\frac{k}{2})} \\
&= 2\pi i^{-k} \mathcal{W}_{0,k}(y, s) 2^{1-2s} (2s-1) \Gamma(2s-1) (4\pi)^{k-1} \nu^{s-k/2} \\
&= 2\pi i^{-k} \mathcal{W}_{0,k}(y, s) 2^{-1+2k-2s} \pi^{k-1} \Gamma(2s) \nu^{s-k/2}.
\end{aligned}$$

□

4. WEAK MAASS-POINCARÉ SERIES IN THE PLUS SPACE

Throughout this section, we assume $k = \lambda + \frac{1}{2}$ where $\lambda \in \mathbb{Z}$ and $N = 4N'$. Following [14, p. 250], we employ Kohnen's projection operator pr_k^+ to construct a weight k Maass-Poincaré series that satisfies the plus space condition, whose Fourier coefficients are supported on $(-1)^\lambda n \equiv 0, 1 \pmod{4}$. The weight k projection operator pr_k^+ is defined by

$$pr_k^+(f) = \frac{1}{2} f + (-1)^k \frac{1}{2\sqrt{2}} \sum_{\nu \pmod{4}} f|_k B \tilde{A}_\nu,$$

where

$$B = \left(\begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix}, e^{(2\lambda+1)\pi i/4} \right) \quad \text{and} \quad A_\nu = \begin{pmatrix} 1 & 0 \\ 4\nu & 1 \end{pmatrix}.$$

For each m satisfying $(-1)^\lambda m \equiv 0, 1 \pmod{4}$ and $\operatorname{Re}(s) > 1$, we define the Poincaré series $F_{m,k,N}^+(z, s)$ by

$$(4.1) \quad F_{m,k,N}^+(z, s) = pr_k^+(F_{m,k,N}(z, s)).$$

The function $F_{m,k,N}^+(z, s)$ has weight k for $\Gamma_0(N)$ and satisfies

$$(4.2) \quad \Delta_k F_{m,k,N}^+(z, s) = \left(s - \frac{k}{2}\right) \left(1 - \frac{k}{2} - s\right) F_{m,k,N}^+(z, s)$$

as $F_{m,k,N}(z, s)$ does. The Fourier expansion of the weakly harmonic Maass form $F_{m,k,4}^+(z, k/2)$ is computed in [8, Theorem 3.5] when $m < 0$ and $k \geq 3/2$. In fact, $F_{m,k,4}^+(z, k/2)$ is a weakly holomorphic modular form if $k > 3/2$ and $m < 0$. Later, the Fourier expansion of $F_{m,k,4}^+(z, 1 - k/2)$ for $k \leq 1/2$ and $m < 0$ is calculated in [3, Theorem 2.1]. Now we establish the Fourier coefficients of $F_{m,k,N}^+(z, s)$ for any m, k, N and s by proving the following three lemmas first.

Lemma 4.1. [14, Proposition 3] *For $\alpha = (\frac{-4}{N'})$, we set*

$$\eta(\frac{-\alpha}{N'}) = \left(\begin{pmatrix} 1 & 0 \\ -\alpha N' & 1 \end{pmatrix}, (-\alpha N' z + 1)^k \right) \text{ and } \eta(\frac{1}{2N'}) = \left(\begin{pmatrix} 1 & 0 \\ 2N' & 1 \end{pmatrix}, (2N' z + 1)^k \right).$$

For a weight k weakly harmonic Maass form g on $\Gamma_0(N)$, if we write

$$\begin{aligned} g(z) &= \sum_n a(n, y) q^n \\ g|_k \eta(\frac{-4}{N'}) (z) &= \sum_n a(\frac{-4}{N'}) (n, y) q^{n/4} \\ g|_k \eta(\frac{1}{2N'}) (z) &= \sum_{(-1)^\lambda n \equiv 1(4)} a(\frac{1}{2N'}) (n, y) q^{n/4}, \end{aligned}$$

then

$$\begin{aligned} pr_k^+(g)(z) &= \sum_{n \equiv 0(4)} \left(a(n, y) + (1 - (-1)^\lambda i) 2^{2\lambda-1} i^{n/4} a^{(-\alpha/N')} \left(\frac{n}{4}, 16y \right) \right) q^n \\ &+ \sum_{(-1)^\lambda n \equiv 1(4)} \left(a(n, y) + 2^{\lambda-1} \left(\frac{(-1)^\lambda n}{2} \right) a^{(1/N')} (n, 4y) \right) q^n. \end{aligned}$$

The original [14, Proposition 3] is for a cusp form g , but the result still holds for a weakly harmonic Maass form g . As Fourier coefficients of $F := F_{m,k,N}(z, s)$ are already computed in Theorem 3.2, we compute those for $F|_k \eta(\frac{-4}{N'}) (z, s)$ and for $F|_k \eta(\frac{1}{2N'}) (z, s)$ in two lemmas below to find the Fourier expansion of $F_{m,k,N}^+(z, s)$.

Lemma 4.2. *If m is an integer, then $F|_k \eta(\frac{-4}{N'}) (z, s)$ has the Fourier expansion*

$$F_{m,k,N}|_k \eta(\frac{-4}{N'}) (z, s) = \sum_{n \in \mathbb{Z}} c'_{m,k}(n, s) \mathcal{W}_{n,k} \left(\frac{y}{4}, s \right) e \left(\frac{nx}{4} \right),$$

where the coefficients $c'_{m,k}(n, s)$ are given by

$$2\pi i^{-k} \frac{(-1)^{\lambda_i}}{4} i^{-n} \sum_{\substack{c>0 \\ c \equiv 1(2)}} H\left(\frac{m}{4}, n\right) \times \begin{cases} |4mn|^{\frac{1-k}{2}} J_{2s-1}\left(\frac{4\pi\sqrt{|4mn|}}{Nc}\right), & mn > 0, \\ |4mn|^{\frac{1-k}{2}} I_{2s-1}\left(\frac{4\pi\sqrt{|4mn|}}{Nc}\right), & mn < 0, \\ 2^{k-1}\pi^{s+\frac{k}{2}-1}|4(m+n)|^{s-\frac{k}{2}}(Nc)^{1-2s}, & mn = 0, m+n \neq 0, \\ 2^{2k-2}\pi^{k-1}\Gamma(2s)(2Nc)^{1-2s}4^{s-k/2}, & m = n = 0, \end{cases}$$

where $H\left(\frac{m}{4}, n\right) = \left(\frac{4}{-N'c}\right) \left(\frac{-4}{N'c}\right)^{-k} \sum_{\delta(N'c)^*} \left(\frac{\delta}{N'c}\right) e\left(\frac{n\delta+m\delta^{-1}}{N'c}\right)$.

Proof. By the exactly same argument in the proof of Proposition 4 of [14] again, we get

$$\begin{aligned} F|_k \eta^{\left(\frac{-4}{N'}\right)}(z, s) &= \sum_{c>0, c \equiv 1(2)} \left(\frac{-4}{-N'c}\right)^k \sum_{d(Nc)^*, d \equiv -N'c(4)} \left(\frac{Nc}{d}\right) \\ &\quad \times \sum_{r \in \mathbb{Z}} \left(Nc\left(\frac{z}{4} + r\right) + d\right)^{-k} \varphi_{m,k}\left(4\frac{a\left(\frac{z}{4} + r\right) + \frac{b}{4}}{Nc\left(\frac{z}{4} + r\right) + d}, s\right). \end{aligned}$$

The Fourier expansion of the inner sum is given by Lemma 3.7 and we see that

$$\begin{aligned} &\frac{1}{Nc} \left(\frac{-4}{-N'c}\right)^k \sum_{d(Nc)^*, d \equiv -N'c(4)} \left(\frac{Nc}{d}\right) e\left(\frac{nd+4md^{-1}}{Nc}\right) \\ &= \frac{(-1)^{\lambda_i}}{4} i^{-n} \left(\frac{4}{-N'c}\right) \left(\frac{-4}{N'c}\right)^{-k} \sum_{d(N'c)^*} \left(\frac{d}{N'c}\right) e\left(\frac{nd+4^{-1}md^{-1}}{N'c}\right) \\ &= \frac{(-1)^{\lambda_i}}{4} i^{-n} H(m/4, n), \end{aligned}$$

from which Lemma 4.2 follows. □

Lemma 4.3. *If m is an integer, then $F|_k \eta^{\left(\frac{1}{2N'}\right)}(z, s)$ has the Fourier expansion*

$$F_{m,k,N}|_k \eta^{\left(\frac{1}{2N'}\right)}(z, s) = \sum_{n \in \mathbb{Z}} c''_{m,k}(n, s) \mathcal{W}_{n,k}\left(\frac{y}{4}, s\right) e\left(\frac{nx}{4}\right),$$

where the coefficients $c''_{m,k}(n, s)$ are given by

$$2\pi i^{-k} \sum_{\substack{c>0 \\ c \equiv 1(2)}} \frac{K_k(4m, n, 2Nc)}{2Nc} \times \begin{cases} |4mn|^{\frac{1-k}{2}} J_{2s-1}\left(\frac{4\pi\sqrt{|4mn|}}{2Nc}\right), & mn > 0, \\ |4mn|^{\frac{1-k}{2}} I_{2s-1}\left(\frac{4\pi\sqrt{|4mn|}}{2Nc}\right), & mn < 0, \\ 2^{k-1}\pi^{s+\frac{k}{2}-1}|4(m+n)|^{s-\frac{k}{2}}(2Nc)^{1-2s}, & mn = 0, m+n \neq 0, \\ 2^{2k-2}\pi^{k-1}\Gamma(2s)(4Nc)^{1-2s}4^{s-k/2}, & m = n = 0. \end{cases}$$

Proof. Arguing analogously as in the proof of Lemman 4.2, we obtain

$$\begin{aligned} F|_k \eta^{(\frac{1}{2N})}(z, s) &= \sum_{c>0, c \equiv 1(2)} \sum_{d(2Nc)^*} \left(\frac{2Nc}{d} \right) \left(\frac{-4}{d} \right)^k \\ &\quad \times \sum_{r \in \mathbb{Z}} \left(2Nc \left(\frac{z}{4} + r \right) + d \right)^{-k} \varphi_{m,k} \left(4 \frac{a \left(\frac{z}{4} + r \right) + \frac{b}{4}}{2Nc \left(\frac{z}{4} + r \right) + d}, s \right). \end{aligned}$$

Now Lemma 4.3 follows from Lemma 3.7. \square

Applying Lemma 4.1 together with Theorem 3.2, Lemma 4.2 and Lemma 4.3, we have

Theorem 4.4. *For any m and s satisfying $(-1)^\lambda m \equiv 0, 1 \pmod{4}$ and $\operatorname{Re}(s) > 1$, $F_{m,k,N}^+(z, s)$ has the Fourier expansion*

$$F_{m,k,N}^+(z, s) = \mathcal{M}_{m,k}(y, s)e(mx) + \sum_{(-1)^\lambda n \equiv 0, 1(4)} b_{m,k}(n, s) \mathcal{W}_{n,k}(y, s)e(nx),$$

where the coefficients $b_{m,k}(n, s)$ are given by

$$2\pi i^{-k} \sum_{c>0} \left(1 + \left(\frac{4}{N'c} \right) \right) \frac{K_k(m, n, Nc)}{Nc} \times \begin{cases} |mn|^{\frac{1-k}{2}} J_{2s-1} \left(\frac{4\pi\sqrt{|mn|}}{Nc} \right), & mn > 0, \\ |mn|^{\frac{1-k}{2}} I_{2s-1} \left(\frac{4\pi\sqrt{|mn|}}{Nc} \right), & mn < 0, \\ 2^{k-1} \pi^{s+\frac{k}{2}-1} |m+n|^{s-\frac{k}{2}} (Nc)^{1-2s}, & mn = 0, m+n \neq 0, \\ 2^{2k-2} \pi^{k-1} \Gamma(2s) (2Nc)^{1-2s}, & m = n = 0. \end{cases}$$

Proof. If $n \equiv 0 \pmod{4}$, then we find that

$$\left(1 + \left(\frac{4}{N'c} \right) \right) K_k(m, n, Nc) = \begin{cases} K_k(m, n, Nc), & c \equiv 0(2), \\ K_k(m, n, Nc) + (1 - (-1)^\lambda i) \frac{(-1)^\lambda i}{4} H \left(\frac{m}{4}, \frac{n}{4} \right), & c \equiv 1(2). \end{cases}$$

But if $(-1)^\lambda n \equiv 1 \pmod{4}$, then we have

$$\left(1 + \left(\frac{4}{N'c} \right) \right) K_k(m, n, Nc) = \begin{cases} K_k(m, n, Nc), & c \equiv 0(2), \\ K_k(m, n, Nc) + \frac{1}{\sqrt{2}} \frac{(-1)^\lambda i}{4} K_k(4m, n, 2Nc), & c \equiv 1(2). \end{cases}$$

Theorem now follows from Lemma 4.1, Theorem 3.2, Lemma 4.2 and Lemma 4.3 altogether. \square

5. WEIGHT 3/2 WEAKLY HARMONIC MAASS FORMS

By the theory of the resolvent kernel again, Poincaré series $F_{m,k,N}^+(z, s)$ has an analytic continuation in s to $\operatorname{Re}(s) > 1/2$ except for possibly finitely many simple poles in $(1/2, 1)$. In fact, for $d \equiv 0, 1 \pmod{4}$ and $D \equiv 0, 3 \pmod{4}$,

$$(5.1) \quad F_{d,1/2,4}^+(z, s) = \mathcal{M}_{d,1/2}(y, s)e(dx) + \sum_{-D \equiv 0, 1(4)} b_{d,1/2}(-D, s) \mathcal{W}_{-D,1/2}(y, s)e(-Dx)$$

has a simple pole at $s = 3/4$. Hence Duke, Imamoglu and Tóth [9] used a weak Maass form

$$(5.2) \quad h_{d,1/2}(z, s) := F_{d,1/2,4}^+(z, s) - \frac{b_{d,1/2}(0, s)}{b_{0,1/2}(0, s)} F_{0,1/2,4}^+(z, s) \quad (d \neq 0)$$

that is holomorphic at $s = 3/4$ and set $h_{d,1/2}(z) := h_{d,1/2}(z, 3/4)$ if $d \neq 0$ and $h_{0,1/2}(z) := f_0(z)$ so that $h_{d,1/2}(z)$ form a basis for $H_{1/2}^!$. Furthermore, they proved that $h_{d,1/2}(z) = f_d(z)$ if $d < 0$ and $\xi_{1/2}(h_{d,1/2}(z)) = -2\sqrt{d}g_{-d}(z) \in M_{3/2}^!$ if $d > 0$ [9, Proposition 1].

It requires a slightly different approach to construct a basis for $H_{3/2}^!$. Let $-D \equiv 0, 1 \pmod{4}$ and $F_D^+(z, s) := F_{D,3/2,4}^+(z, s)$. Then by Theorem 4.4, the Fourier expansion of $F_D^+(z, s)$ is given by

$$(5.3) \quad F_D^+(z, s) = \mathcal{M}_{D,3/2}(y, s)e(Dx) + \sum_{-d \equiv 0,3 \pmod{4}} b_{D,3/2}(-d, s) \mathcal{W}_{-d,3/2}(y, s)e(-dx).$$

First, note from (2.21) that

$$(5.4) \quad \lim_{s \rightarrow 3/4} \mathcal{M}_{D,3/2}(y, s)e(Dx) = \begin{cases} \frac{2}{\sqrt{\pi}} q^D, & D \neq 0, \\ 1, & D = 0. \end{cases}$$

Also, it follows from (2.14), (2.20), and (2.11) that

$$(5.5) \quad \lim_{s \rightarrow 3/4} \mathcal{W}_{-d,3/2}(y, s) = \begin{cases} \frac{2}{\sqrt{\pi}} |d|^{\frac{1}{2}} e^{2\pi dy} & d < 0, \\ \lim_{s \rightarrow 3/4} \Gamma(s - \frac{3}{4})^{-1} d^{1/2} e^{2\pi dy} \Gamma(-\frac{1}{2}, 4\pi dy), & d > 0, \\ \lim_{s \rightarrow 3/4} \Gamma(s - \frac{3}{4})^{-1} \frac{2}{\pi \sqrt{y}}, & d = 0. \end{cases}$$

Now recall the symmetric property of the Kloosterman sum ([3, Proposition 3.1])

$$K_{\frac{3}{2}}(m, n, c) = K_{\frac{3}{2}}(n, m, c) = -iK_{\frac{1}{2}}(-m, -n, c),$$

which implies

$$(5.6) \quad b_{D,3/2}(-d, s) = b_{-D,1/2}(d, s) \times \begin{cases} -|Dd|^{-\frac{1}{2}}, & Dd \neq 0, \\ -2\pi^{\frac{1}{2}} |D - d|^{-\frac{1}{2}}, & Dd = 0, D - d \neq 0, \\ -4\pi, & D = d = 0. \end{cases}$$

It is proved in [9, Lemma 3, (2.25), (2.26)] that $b_{-D,1/2}(d, s)$ has a simple pole at $s = \frac{3}{4}$ if and only if both d and $-D$ are equal to zero or non-zero squares and

$$(5.7) \quad \text{Res}_{s=3/4} b_{-D,1/2}(d, s) = \begin{cases} \frac{3}{16\pi}, & D = 0, d = 0, \\ \frac{3m}{2\pi}, & D = 0, d = m^2 \neq 0, \\ \frac{12m\sqrt{-D}}{\pi}, & -D = \square \neq 0, d = m^2 \neq 0. \end{cases}$$

Hence the coefficient $b_{D,3/2}(-d, s)$ has a simple pole at $s = \frac{3}{4}$ if and only if both $-D$ and d are equal to zero or non-zero squares as well, but in which case the singularity cancels with the

zero of $\Gamma(s - 3/4)^{-1}$ appearing in $\mathcal{W}_{-d,3/2}(y, s)$. Therefore $F_D^+(z, s)$ is holomorphic in s . More precisely, if $-D$ is a non-zero square, then it follows from (5.5), (5.6) and (5.7) that

$$\begin{aligned}
 & \lim_{s \rightarrow 3/4} b_{D,3/2}(-d, s) \mathcal{W}_{-d,3/2}(y, s) \\
 &= \begin{cases} (\text{Res}_{s=3/4} b_{D,3/2}(-m^2, s)) |m| e^{2\pi m^2 y} \Gamma(-\frac{1}{2}, 4\pi m^2 y), & d = m^2 \neq 0, \\ (\text{Res}_{s=3/4} b_{D,3/2}(0, s)) \frac{2}{\pi \sqrt{y}}, & d = 0, \end{cases} \\
 &= \begin{cases} -\text{Res}_{s=3/4} b_{-D,1/2}(m^2, s) |D|^{-1/2} e^{2\pi m^2 y} \Gamma(-\frac{1}{2}, 4\pi m^2 y), & d = m^2 \neq 0, \\ -\text{Res}_{s=3/4} b_{-D,1/2}(0, s) \frac{4}{\sqrt{\pi |D| y}}, & d = 0, \end{cases} \\
 (5.8) \quad &= \begin{cases} -\frac{12m}{\pi} e^{2\pi m^2 y} \Gamma(-\frac{1}{2}, 4\pi m^2 y), & d = m^2 \neq 0, \\ -\pi^{-3/2} \frac{6}{\sqrt{y}}, & d = 0, \end{cases}
 \end{aligned}$$

where the last is obtained from the fact $b_{0,1/2}(d, s) = b_{d,1/2}(0, s)$. Similarly, if $D = 0$, then we have

$$(5.9) \quad \lim_{s \rightarrow 3/4} b_{0,3/2}(-d, s) \mathcal{W}_{-d,3/2}(y, s) = \begin{cases} -\frac{3m}{\sqrt{\pi}} e^{2\pi m^2 y} \Gamma(-\frac{1}{2}, 4\pi m^2 y), & d = m^2 \neq 0, \\ -\frac{3}{2\pi \sqrt{y}}, & d = 0. \end{cases}$$

Thus we reach the following result.

Proposition 5.1. *Let $D \equiv 0, 3 \pmod{4}$.*

- (1) *If D is positive, then $F_D^+(z, 3/4) = 0$.*
- (2) *If D is negative and $-D$ is not a square, then*

$$F_D^+(z, 3/4) \in M_{3/2}^!.$$

- (3) *If D is negative and $-D$ is a non-zero square, then*

$$F_D^+(z, 3/4) - \frac{4}{\sqrt{\pi}} F_0^+(z, 3/4) \in M_{3/2}^!.$$

Proof. Suppose $D > 0$ or $D < 0$ but $-D \neq \square$. Then there is no pole at $s = 3/4$ and

$$(5.10) \quad b_{D,3/2}(-d, 3/4) \mathcal{W}_{-d,3/2}(y, 3/4) e(-dx) = \begin{cases} \frac{2}{\sqrt{\pi}} b_{D,3/2}(-d, 3/4) |d|^{1/2} q^{-d}, & d < 0; \\ 0, & d \geq 0 \end{cases}$$

which yields

$$F_D^+(z, 3/4) = \frac{2}{\sqrt{\pi}} \left(q^D + \sum_{0 < -d \equiv 0, 3 \pmod{4}} b_{D,3/2}(-d, 3/4) |d|^{1/2} q^{-d} \right).$$

This proves (2) and (1) as well, because there is no weight $3/2$ cusp form satisfying the plus space condition. For $-D = \square$, it follows from (5.4), (5.5), (5.8), (5.9) and (5.10) that

$$(5.11) \quad F_D^+(z, 3/4) = \frac{2}{\sqrt{\pi}} q^D - \pi^{-3/2} \frac{6}{\sqrt{y}} + \sum_{0 < -d \equiv 0, 3 \pmod{4}} \frac{2}{\sqrt{\pi}} b_{D, 3/2}(-d, 3/4) |d|^{1/2} q^{-d} - \frac{12}{\pi} \sum_{0 < d = \square} \sqrt{d} \Gamma\left(-\frac{1}{2}, 4\pi|d|y\right) q^{-d} \quad \text{if } 0 \neq -D = \square$$

and

$$(5.12) \quad F_0^+(z, 3/4) = 1 - \frac{3}{2\pi\sqrt{y}} + \sum_{0 < -d \equiv 0, 3 \pmod{4}} \frac{2}{\sqrt{\pi}} b_{0, 3/2}(-d, 3/4) |d|^{1/2} q^{-d} - \frac{3}{\sqrt{\pi}} \sum_{0 < d = \square} \sqrt{d} \Gamma\left(-\frac{1}{2}, 4\pi|d|y\right) q^{-d}, \quad \text{if } D = 0,$$

which proves (3). Alternatively, (3) can be proved by showing $\xi_{3/2}(F_D^+(z, 3/4) - \frac{4}{\sqrt{\pi}} F_0^+(z, 3/4)) = 0$. \square

The Poincaré series $F_0^+(z, 3/4) = -12E(z)$, where $E(z)$ is Zagier's Eisenstein series of weight $3/2$ for $\Gamma_0(4)$ ([20]) and

$$E(z) = \sum_{n=0}^{\infty} H(n) q^n + \frac{1}{16\pi\sqrt{y}} \sum_{n=-\infty}^{\infty} \beta(4\pi n^2 y) q^{-n^2}.$$

Here $H(n)$ is the Hurwitz class number with $H(0) = \zeta(-1) = -\frac{1}{12}$ and $\beta(s) = \int_1^{\infty} t^{-3/2} e^{-st} dt$. This yields for positive integer n ,

$$(5.13) \quad H(n) = -\frac{1}{6} \sqrt{\frac{n}{\pi}} b_{0, 3/2}(n, 3/4).$$

The results (2) and (3) of Proposition 5.1 were proved in [8, Proposition 3.6] in a slightly different form. The series in (3) reflects a symmetric relation with the weight $1/2$ weakly harmonic Maass form in (5.2) as

$$F_D^+(z, 3/4) - \frac{4}{\sqrt{\pi}} F_0^+(z, 3/4) = \lim_{s \rightarrow 3/4} \left(F_D^+(z, s) - \frac{b_{D, 3/2}(0, s)}{b_{0, 3/2}(0, s)} F_0^+(z, s) \right).$$

In order to construct a nontrivial weakly harmonic Maass form for the case $D > 0$, we consider the derivative of the Poincaré series $F_D^+(z, s)$ at $s = 3/4$.

Proposition 5.2. *Let $0 < D \equiv 0, 3 \pmod{4}$. Then*

$$(5.14) \quad \begin{aligned} \frac{\partial}{\partial s} F_D^+(z, s)|_{s=3/4} &= -12 \frac{H(D)}{\sqrt{\pi D y}} - i \Gamma\left(-\frac{1}{2}, -4\pi D y\right) q^D - 2\sqrt{\pi} i q^D \\ &+ \sum_{\substack{0 < -d \equiv 0, 3 \pmod{4} \\ -d \neq D}} \frac{\partial}{\partial s} b_{D, 3/2}(-d, s)|_{s=3/4} \frac{2\sqrt{|d|}}{\sqrt{\pi}} q^{-d} \\ &+ \sum_{0 < d \equiv 0, 1 \pmod{4}} b_{D, 3/2}(-d, 3/4) \sqrt{|d|} \Gamma\left(-\frac{1}{2}, 4\pi d y\right) q^{-d}. \end{aligned}$$

Proof. Applying (5.3), (5.5) and Proposition 5.1(1), we have

$$\begin{aligned} \frac{\partial}{\partial s} F_D^+(z, s)|_{s=\frac{3}{4}} &= \mathcal{X}(y)e(Dx) + \sum_{\substack{n>0 \\ n \neq D}} \frac{\partial}{\partial s} b_{D,3/2}(n, s)|_{s=\frac{3}{4}} \frac{2\sqrt{n}}{\sqrt{\pi}} q^n \\ &\quad + \sum_{n \leq 0} b_{D,3/2}(n, 3/4) \frac{\partial}{\partial s} \mathcal{W}_{n,3/2}(y, s)|_{s=\frac{3}{4}} e(nx). \end{aligned}$$

where $\mathcal{X}(y) = \frac{\partial}{\partial s} (\mathcal{M}_{D,3/2}(y, s) + b_{D,3/2}(D, s) \mathcal{W}_{D,3/2}(y, s))|_{s=\frac{3}{4}}$. It follows from (2.14), (2.11) and the fact $W_{\mu,\nu} = W_{\mu,-\nu}$ that $\frac{\partial}{\partial s} \mathcal{W}_{n,3/2}(y, s)|_{s=\frac{3}{4}} = \sqrt{|n|} e^{2\pi|n|y} \Gamma(-\frac{1}{2}, 4\pi|n|y)$ if $n < 0$. Also, $\frac{\partial}{\partial s} \mathcal{W}_{0,3/2}(y, s)|_{s=\frac{3}{4}} = \frac{2}{\pi\sqrt{y}}$. Hence

$$\begin{aligned} \frac{\partial}{\partial s} F_D^+(z, s)|_{s=\frac{3}{4}} &= \mathcal{X}(y)e(Dx) + \sum_{\substack{n>0 \\ n \neq D}} \frac{\partial}{\partial s} b_{D,3/2}(n, s)|_{s=\frac{3}{4}} \frac{2\sqrt{n}}{\sqrt{\pi}} q^n \\ (5.15) \quad &\quad + b_{D,3/2}(0, 3/4) \frac{2}{\pi\sqrt{y}} + \sum_{n<0} b_{D,3/2}(n, 3/4) \sqrt{|n|} \Gamma\left(-\frac{1}{2}, 4\pi|n|y\right) q^n. \end{aligned}$$

Note from (5.6) and the fact $h_{d,1/2} = f_d$ for $d < 0$ that $b_{D,3/2}(D, 3/4) = -\frac{1}{D} b_{-D,1/2}(-D, 3/4) = -\frac{1}{\sqrt{D}}$ and $M_{\frac{3}{4},\frac{1}{4}}(y) = W_{\frac{3}{4},\frac{1}{4}}(y) = W_{\frac{3}{4},-\frac{1}{4}}(y) = y^{3/4} e^{-y/2}$. We then have

$$\begin{aligned} \mathcal{X}(y) &= \left[\frac{\partial}{\partial s} \left(\Gamma(2s)^{-1} (4\pi Dy)^{-3/4} M_{\frac{3}{4},s-1/2}(4\pi Dy) - \Gamma(s + \frac{3}{4})^{-1} (4\pi Dy)^{-3/4} W_{\frac{3}{4},s-1/2}(4\pi Dy) \right) \right]_{s=\frac{3}{4}} \\ &= \mathcal{A} + \frac{2}{\sqrt{\pi}} (4\pi Dy)^{-3/4} \mathcal{B}, \end{aligned}$$

where

$$\mathcal{A} = \frac{d}{ds} \left(\Gamma(2s)^{-1} - \Gamma(s + \frac{3}{4})^{-1} \right) (4\pi Dy)^{-3/4} W_{\frac{3}{4},\frac{1}{4}}(4\pi Dy) = -\frac{2}{\sqrt{\pi}} (2 - \gamma - 2 \ln 2) e^{-2\pi Dy}$$

for the Euler constant γ and

$$\begin{aligned} \mathcal{B} &= \left[\frac{\partial}{\partial s} \left(M_{\frac{3}{4},s-1/2}(4\pi Dy) - W_{\frac{3}{4},s-1/2}(4\pi Dy) \right) \right]_{s=\frac{3}{4}} \\ &= \left[\frac{\partial}{\partial s} \left(M_{\frac{3}{4},s-1/2}(4\pi Dy) - \frac{\Gamma(1-2s)}{\Gamma(1/4-s)} M_{\frac{3}{4},s-1/2}(4\pi Dy) - \frac{\Gamma(2s-1)}{\Gamma(s-3/4)} M_{\frac{3}{4},1/2-s}(4\pi Dy) \right) \right]_{s=\frac{3}{4}} \\ &= \left[-\frac{d}{ds} \frac{\Gamma(1-2s)}{\Gamma(1/4-s)} \right]_{s=\frac{3}{4}} M_{\frac{3}{4},\frac{1}{4}}(4\pi Dy) - \left[\frac{d}{ds} \frac{\Gamma(2s-1)}{\Gamma(s-3/4)} \right]_{s=\frac{3}{4}} M_{\frac{3}{4},-\frac{1}{4}}(4\pi Dy) \\ &= (2 - \gamma - 2 \ln 2) M_{\frac{3}{4},\frac{1}{4}}(4\pi Dy) - \sqrt{\pi} M_{\frac{3}{4},-\frac{1}{4}}(4\pi Dy) \\ &= (2 - \gamma - 2 \ln 2) (4\pi Dy)^{3/4} e^{-2\pi Dy} - \sqrt{\pi} M_{\frac{3}{4},-\frac{1}{4}}(4\pi Dy). \end{aligned}$$

The second equality above holds by (2.6). But by (2.2), (2.4), (2.5) and the transformation $M(a, b, z) = e^z M(b - a, b, -z)$,

$$\begin{aligned}
M_{\frac{3}{4}, -\frac{1}{4}}(4\pi Dy) &= e^{-2\pi Dy} (4\pi Dy)^{1/4} M(-\frac{1}{2}, \frac{1}{2}, 4\pi Dy) \\
&= e^{2\pi Dy} (4\pi Dy)^{1/4} M(1, \frac{1}{2}, -4\pi Dy) \\
&= e^{2\pi Dy} (4\pi Dy)^{1/4} \left(\frac{1}{2} U(1, \frac{1}{2}, -4\pi Dy) + \sqrt{\pi} (-4\pi Dy)^{1/2} M(\frac{3}{2}, \frac{3}{2}, -4\pi Dy) \right) \\
&= \frac{1}{2} e^{2\pi Dy} (4\pi Dy)^{1/4} U(1, \frac{1}{2}, -4\pi Dy) + \sqrt{\pi} i (4\pi Dy)^{3/4} e^{-2\pi Dy} \\
&= \frac{i}{2} e^{-2\pi Dy} (4\pi Dy)^{3/4} \Gamma(-\frac{1}{2}, -4\pi Dy) + \sqrt{\pi} i (4\pi Dy)^{3/4} e^{-2\pi Dy},
\end{aligned}$$

where the last line is obtained from $\Gamma(s, z) = e^{-z} z^s U(1, 1 + s, z)$. Hence,

$$\mathcal{X}(y)e(Dx) = -i\Gamma(-\frac{1}{2}, -4\pi Dy)q^D - 2\sqrt{\pi}iq^D,$$

and finally we establish (5.14) by applying (5.6) and (5.13). \square

In general, a harmonic weak Maass form has a Fourier expansion

$$h(z) = \sum_{n \gg -\infty} c_h^+(n)q^n + c_h^-(0)y^{1-k} + \sum_{0 \neq n \ll \infty} c_h^-(n)\Gamma(1-k, -4\pi ny)q^n$$

so that

$$(5.16) \quad \xi_k(h) = (1-k)\overline{c_h^-(0)} - \sum_{0 \neq n \ll \infty} \overline{c_h^-(n)(-4\pi n)^{1-k}}q^{-n}.$$

This implies with (5.14) that

$$\begin{aligned}
(5.17) \quad \xi_{3/2} \left(\frac{\partial}{\partial s} F_D^+(z, s) \Big|_{s=\frac{3}{4}} \right) &= \frac{6H(D)}{\sqrt{\pi D}} + \frac{1}{\sqrt{4\pi D}} q^{-D} - \frac{1}{\sqrt{4\pi}} \sum_{0 < d \equiv 1 \pmod{4}} \overline{b_{D, 3/2}(-d, 3/4)} q^d \\
&= \frac{6H(D)}{\sqrt{\pi D}} + \frac{1}{\sqrt{4\pi D}} \left(q^{-D} + \sum_{0 < d \equiv 1 \pmod{4}} \overline{b_{-D, 1/2}(d, 3/4)} \frac{1}{\sqrt{|d|}} q^d \right).
\end{aligned}$$

Using (5.2), (5.6) and (5.7), we may write the weight 1/2 weakly harmonic Maass form

$$h_{d, 1/2}(z) = \mathcal{M}_{d, 1/2}(y, 3/4)e(dx) - \delta_{\square, d} 8\sqrt{dy} + \sum_{0 \neq n \equiv 0, 1 \pmod{4}} a_d(n) \mathcal{W}_{n, 1/2}(y, 3/4)e(nx),$$

where

$$a_d(n) = b_{d, 1/2}(n, 3/4) - \delta_{\square, n} \frac{4\sqrt{nd}}{\sqrt{\pi}} b_{-d, 3/2}(0, 3/4).$$

Since $h_{d,1/2}(z) = f_d(z)$, which is given in (1.1) if $d < 0$ ([9, Proposition 1]), we can observe using (5.13) that

$$(5.18) \quad \xi_{3/2} \left(\frac{\partial}{\partial s} F_D^+(z, s) \Big|_{s=\frac{3}{4}} \right) = \frac{1}{\sqrt{4\pi D}} f_{-D}(z) + \frac{6H(D)}{\sqrt{\pi D}} \theta(z),$$

where $\theta(z) = \sum_{n \in \mathbb{Z}} q^{n^2}$. We are now ready to construct a basis for the space of weight $3/2$ weakly harmonic Maass forms.

Theorem 5.3. *For each $D \equiv 0, 3 \pmod{4}$, let us define*

$$h_{D,3/2}(z) := \begin{cases} \frac{\partial}{\partial s} F_D^+(z, s) \Big|_{s=3/4} - 8\sqrt{\frac{\pi}{D}} H(D) F_0^+(z, 3/4), & D > 0, \\ F_D^+(z, 3/4) - \frac{4}{\sqrt{\pi}} F_0^+(z, 3/4), & -D = \square, \\ F_D^+(z, 3/4), & D = 0 \text{ or } 0 < -D \neq \square. \end{cases}$$

The set $\{h_{D,3/2} | D \equiv 0, 3 \pmod{4}\}$ forms a basis for $H_{3/2}^!$. When $D < 0$, $h_{D,3/2}(z)$ forms a unique basis for $M_{3/2}^!$ of the form given in (1.2), because

$$h_{D,3/2}(z) = \frac{2}{\sqrt{\pi}} g_D(z).$$

When $D \geq 0$, $h_{D,3/2}(z)$ is a unique weakly harmonic Maass form with bounded holomorphic part satisfying

$$(5.19) \quad \xi_{3/2}(h_{D,3/2}(z)) = \frac{1}{2\sqrt{\pi D}} f_{-D}(z) \quad (D > 0) \quad \text{and} \quad \xi_{3/2}(h_{0,3/2}(z)) = \frac{3}{4\pi} f_0(z),$$

where f_d is given by (1.1).

Proof. As it satisfies

$$(5.20) \quad \Delta_{3/2} \frac{\partial}{\partial s} F_D^+(z, s) = \left(s - \frac{3}{4}\right) \left(\frac{1}{4} - s\right) \frac{\partial}{\partial s} F_D^+(z, s),$$

$h_{D,3/2}(z)$ is clearly a weakly harmonic Maass form for each D . Furthermore, Proposition 5.1 and its proof imply that $h_{D,3/2}(z) = \frac{2}{\sqrt{\pi}} g_D \in M_{3/2}^!$ when $D < 0$ and the weight $3/2$ non-holomorphic Eisenstein series $h_{0,3/2}(z) = -12E(z)$ is a well known weakly harmonic Maass form having

$$(5.21) \quad \xi_{3/2}(h_{0,3/2}(z)) = -12\left(-\frac{1}{16\pi}\right)\theta(z) = \frac{3}{4\pi} f_0(z).$$

For $D > 0$, it follows from the definition of $h_{D,3/2}(z)$, (5.16), (5.18) and (5.21) that

$$\xi_{3/2}(h_{D,3/2}(z)) = \frac{1}{2\sqrt{\pi D}} f_{-D}(z).$$

Finally, the uniqueness is a consequence of the fact that the only holomorphic modular form of weight $3/2$ satisfying the plus space condition is zero. \square

Proof of Theorem 1.1. We take the holomorphic part of $2\sqrt{\pi D} h_{D,3/2}(z)$ as $g_D(z)$ if $D > 0$ and the holomorphic part of $\frac{4\pi}{3} h_{0,3/2}(z)$ as $g_0(z)$. Then Theorem 1.1 follows immediately from Theorem 5.3. \square

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